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Wick quantization of cotangent bundles over Riemannian manifolds

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Abstract

A simple geometric procedure is proposed for constructing Wick symbols on cotangent bundles of Riemannian manifolds. The main ingredient of the construction is a method of endowing the cotangent bundle with a formal Kähler structure. The formality means that the metric is lifted from the Riemannian manifold $\mathcal{Q}$ to its phase space $T^*\mathcal{Q}$ in the form of formal power series in momenta with the coefficients being tensor fields on the base. The corresponding Kähler two-form on the total space of $T^*\mathcal{Q}$ coincides with the canonical symplectic form, while the canonical projection of the Kähler metric on the base manifold reproduces the original metric. Some examples are considered, including constant curvature space and nonlinear sigma-models, illustrating the general construction.

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1. Introduction

The word “quantization” usually means a procedure of constructing a quantum mechanical system for a given classical one. This construction is known to involve a great amount of ambiguity and, in this sense, the quantum mechanics provides a more refined description of physical systems than the classical mechanics. Mathematically, this deeper level of description manifests itself in extra geometric structures to be defined on the phase space of a system to perform its consistent quantization. In the framework of the deformation quantization, for instance, a symplectic connection, being not involved in the classical dynamics, becomes a key ingredient of the theory at the quantum level [1,2]. Special types of the deformation quantization can involve more “rigid” geometrical structures (like metric, torsion, complex structure, etc.) and different choices for these structures may lead to inequivalent quantizations of the same phase space.

In this paper we address the question of the Wick deformation quantization on cotangent bundles of Riemannian manifolds. Several reasons can be mentioned motivating this study:

(i) The phase space of most physical systems has the form of cotangent bundle $T^*\mathcal{Q}$ over the configuration space $\mathcal{Q}$. Often, the latter carries a natural (pseudo-)Riemannian metric entering the very formulation of the classical model (point particle in the General Relativity, nonlinear sigma-models, etc.). Even when such a metric is not explicitly involved at the classical level it may appear (sometimes implicitly) upon quantization. For example, the configuration space metric is a basic ingredient of the unique effective action construction [3,4], the most advanced path-integral quantization method known in the field theory. In the series of papers [5–7], the relevance of phase-space metric was argued to the regularization of phase-space path integrals, including those modelling transition amplitudes in quantum gravity [8].

(ii) It is the Wick symbol algebra of physical observables (the Bargmann–Fock representation) which is commonly accepted to serve as a basis for the quantum mechanical description of fields. Usually, the Wick symbols are known but at the level of free fields, that suggests the perturbative treatment for the interaction from the outset. Such a disintegration of the entire theory into the (linear) “free part” and the (nonlinear) “interaction” may happen to be inadequate to the physics, as it can break, for example, fundamental symmetries of the classical model. The nonlinear sigma-models and, in particular, strings on the AdS space are typical examples where no physically reasonable linear approximation can be found for the phase/configuration space geometry. Thus, to get a quantum mechanical description respecting the geometry of such essentially nonlinear models, like just mentioned, one needs an explicitly covariant and globally defined procedure of Wick quantization for cotangent bundles.

(iii) Finally, the construction of Wick symbols is intimately connected with the metrization of a phase space to be quantized. In fact, whenever all the conditions on the metric, imposed by the Wick quantization, are satisfied, the phase space is proved to be the Kähler manifold [10]. The bundle structure of the phase space $T^*\mathcal{Q}$ allows one to endow the configuration space $\mathcal{Q}$ with the Riemannian metric induced by the canonical embedding $\mathcal{Q} \subset T^*\mathcal{Q}$.

The main idea of this paper is, in a sense, an inversion of the last remark. Starting with the Riemannian metric g on the configuration space $\mathcal{Q}$, we construct a *formal* Kähler metric

$$G = G_{\imath\bar{\jmath}}\,\mathrm{d}z^{\imath}\,\mathrm{d}\bar{z}^{\jmath} = G_{ij}(x,p)\,\mathrm{d}x^i\,\mathrm{d}x^j + G_i^j(x,p)\,\mathrm{d}x^i\,\mathrm{d}p_j + G^{ij}(x,p)\,\mathrm{d}p_i\,\mathrm{d}p_j$$

on the phase space $T^*\mathcal{Q}$ requiring the corresponding Kähler two-form to coincide with the canonical symplectic structure on $T^*\mathcal{Q}$

$$\omega = G_{\imath\bar{\jmath}}\,\mathrm{d}z^{\imath} \wedge \mathrm{d}\bar{z}^{\jmath} = \mathrm{d}p_i \wedge \mathrm{d}x^i.$$

Here $(z^{\imath}, \bar{z}^{\jmath})$ are the complex coordinates adapted to the Kähler structure, while (x^i, p_j) are the canonical coordinates on the cotangent bundle. In this context, the formality means that the components of the metric tensor G are given by formal power series in the momenta p_i. When $(\mathcal{Q}, g)$ is a real-analytical Riemannian manifold, these series can be shown to converge in a tubular neighborhood of $\mathcal{Q} \subset T^*\mathcal{Q}$. In this paper we construct a natural phase-space metric G satisfying boundary condition $G|_{\mathcal{Q}} = g$ and equipping $T^*\mathcal{Q}$ with an integrable Kähler structure. Such a lift of the metric from the base manifold to the cotangent bundle automatically endows the latter with a (formal) connection respecting both the canonical Poisson bracket and the phase-space metric.

Once the Kähler structure is defined, known deformation quantization methods can be immediately applied to build up the algebra of Wick symbols [9–11].

The Weyl deformation quantization of cotangent bundles has been already studied in [12,13]. In those papers an affine connection has been lifted from the configuration space $\mathcal{Q}$ to the so-called homogeneous connection on the phase space $T^*\mathcal{Q}$ which is used then to apply the machinery of the Fedosov deformation quantization [1,2]. In fact, the requirement of homogeneity restricts the components of the connection to be at most linear in momenta. The Wick quantization, as we will see in this paper, implies a different lift for the connection (as it has to respect the metric), leading in general to infinite series in momenta for the lifted connection.

The paper is organized as follows. For reader's convenience, in Section 2 we collect some basic notions and formulae related to the Kähler geometry. In Section 3 a general covariant procedure is proposed for the lift of a Riemannian metric on $\mathcal{Q}$ to a metric on $T^*\mathcal{Q}$ equipping the latter with a Kähler structure. This procedure is quite similar to the construction of a formal exponential map of [14] inspired, in turn, by the "flattering" procedure for the quantum connection in the Fedosov deformation quantization. In all the cases one deals with an iterative construction of formal power series in the fiber coordinates. In Section 4 a set of holomorphic coordinates compatible with the Kähler structure is obtained via the exponential map generated by a Hamiltonian flow. The construction makes it possible to prove the convergence of the formal series for the lifted metric in a tubular neighborhood of the base $\mathcal{Q} \subset T^*\mathcal{Q}$, provided $(\mathcal{Q}, g)$ is real-analytical. In Section 5 we calculate the first Chern class of the Kähler structure which turns out to be zero. In Section 6 the general method is applied to several examples of physical interest including those related to the quantum field theory. In Section 7 we briefly discuss the issues of equivalence between Wick and Weyl quantization, and problems of applying the proposed technique to the field/string theory models.

2. Kähler manifolds

In this section we briefly recall some basic definitions and facts concerning the geometry of (almost-)Kähler manifolds. For more details see, for example, [15].

An almost-Kähler manifold $(\mathcal{M}, J, \omega)$ is a real manifold $\mathcal{M}$ of even dimension together with an almost-complex structure J and a symplectic form ω which are compatible in the following sense:

$$\omega(JX, JY) = \omega(X, Y) \tag{2.1}$$

for any vector fields X, Y. In other words, the smooth field of automorphisms

$$J : T\mathcal{M} \to T\mathcal{M}, \quad J^2 = -1 \tag{2.2}$$

is a canonical transformation of the tangent bundle w.r.t. the symplectic structure ω, and $G(X, Y) = \omega(JX, Y)$ is J-invariant (pseudo-)Riemannian metric on $\mathcal{M}$.

The almost-complex structure J splits the complexified tangent bundle $T^{\mathbb{C}}\mathcal{M}$ into two transverse mutually conjugated subbundles: $T^{\mathbb{C}}\mathcal{M} = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}$, such that

$$J_pX = \mathrm{i}X, \quad \forall X \in T_p^{(1,0)}\mathcal{M}, \qquad J_pY = -\mathrm{i}Y, \quad \forall Y \in T_p^{(0,1)}\mathcal{M} \tag{2.3}$$

for every $p \in \mathcal{M}$. In the natural frame $\{\partial_a = \partial/\partial x^a\}$ and co-frame $\{\mathrm{d}x^a\}$ associated to local coordinates $\{x^a\}$ on $\mathcal{M}$ we have $J^a{}_b = J(\mathrm{d}x^a, \partial_b)$, $\omega_{ab} = \omega(\partial_a, \partial_b)$, $G_{ab} = G(\partial_a, \partial_b)$, and

$$J^a{}_b = G^{ac}\omega_{cb} = G_{bc}\omega^{ca}, \quad a, b, c = 1, \ldots, \dim \mathcal{M}, \tag{2.4}$$

where (G^{ab}) and (ω^{ab}) are the inverse matrices to (G_{ab}) and (ω_{ab}), respectively. It is known that any symplectic manifold $(\mathcal{M}, \omega)$ admits a compatible almost-complex structure J turning $\mathcal{M}$ into an almost-Kähler manifold.

Alternatively, an almost-Kähler manifold can be defined as a pair $(\mathcal{M}, \Lambda)$ in which $\mathcal{M}$ is a real manifold of even dimension equipped with a degenerate Hermitian form Λ, such that

$$\operatorname{rank} \Lambda = \tfrac{1}{2} \dim \mathcal{M}, \quad \det(\operatorname{Im} \Lambda) \neq 0. \tag{2.5}$$

The equivalence of both definitions is set by the formula

$$\Lambda_{ab} = G_{ab} + \mathrm{i}\omega_{ab}. \tag{2.6}$$

Clearly, the subbundles $T^{(0,1)}\mathcal{M}$ and $T^{(1,0)}\mathcal{M}$ of the complexified tangent bundle $T^{\mathbb{C}}\mathcal{M}$ are nothing but the right/left kernel distributions of the form Λ. Note also that the left and right null-vectors of Λ can be obtained from each other by the complex conjugation of their components.

An almost-Kähler manifold $(\mathcal{M}, J, \omega)$ is said to be a Kähler manifold if $T^{(0,1)}\mathcal{M}$ and $T^{(1,0)}\mathcal{M}$ are integrable distributions. In this case there exists an atlas of charts with complex coordinates $\{z^a\}$ and holomorphic transition functions in which terms the Hermitian form Λ takes the block form

$$\Lambda = \begin{pmatrix} G_{m\bar{n}} & | & 0 \\ - & | & - \\ 0 & | & 0 \end{pmatrix}, \quad m, \bar{n} = 1, \ldots, \tfrac{1}{2}\dim\mathcal{M}. \tag{2.7}$$

From the view point of symplectic geometry, the integrable holomorphic/anti-holomorphic distributions $T^{(1,0)}\mathcal{M}$ and $T^{(0,1)}\mathcal{M}$ define a pair of transverse Lagrangian polarizations of $\mathcal{M}$, i.e. $\omega|_{T^{(1,0)}\mathcal{M}} = \omega|_{T^{(0,1)}\mathcal{M}} = 0$. The existence of such polarizations is of primary importance for the physical applications as it makes possible to construct the Hilbert space of states of a quantum-mechanical system associated to the phase space $(\mathcal{M}, \omega)$.

3. The formal Kähler metric construction for cotangent bundles

Let $(\mathcal{Q}, g)$ be a (pseudo-)Riemannian manifold and let ∇ be a compatible symmetric connection. For a coordinate chart $(U, \{x^i\})$, $i = 1, \ldots, \dim\mathcal{Q}$, denote by $\{p_i\}$ the linear coordinates on the fibers of T^*U with respect to the natural frame $\{\mathrm{d}x^i\}$. There is a natural lift of the Riemannian metric g on $\mathcal{Q}$ to that on the total space of the cotangent bundle $T^*\mathcal{Q}$. It is given by

$$G^{(0)} = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j + g^{ij} Dp_i \otimes Dp_j, \tag{3.1}$$

where

$$g^{ik}g_{kj} = \delta^i_j, \qquad D = \mathrm{d}p_i \frac{\partial}{\partial p_i} - \mathrm{d}x^i \nabla_i, \qquad \nabla_i = \frac{\partial}{\partial x^i} + p_k \Gamma^k_{ij} \frac{\partial}{\partial p_j}$$

and Γ^k_{ij} are Cristoffel symbols. Clearly, $(T^*\mathcal{Q}, G^{(0)})$ is a Riemannian manifold. Also, $T^*\mathcal{Q}$ is a symplectic manifold w.r.t. the canonical symplectic structure

$$\omega = \mathrm{d}\theta = Dp_i \wedge \mathrm{d}x^i = \mathrm{d}p_i \wedge \mathrm{d}x^i, \quad \theta = p_i\,\mathrm{d}x^i. \tag{3.2}$$

Both these structures on $T^*\mathcal{Q}$ can be arranged into the Hermitian form

$$\Lambda^{(0)} = G^{(0)} + \mathrm{i}\omega, \quad (\Lambda^{(0)})^\dagger = \Lambda^{(0)}. \tag{3.3}$$

Proposition 1. *The Hermitian form $\Lambda^{(0)}$ endows $T^*\mathcal{Q}$ with the structure of an almost-Kähler manifold. The corresponding almost-complex structure J is integrable iff the Riemannian manifold $(\mathcal{Q}, g)$ is flat.*

Proof. One can check that in each coordinate chart T^*U the local vector fields

$$\mathrm{Vect}(T^*U) \ni V_i^{(0)} = \nabla_i - \mathrm{i} g_{ij} \frac{\partial}{\partial p_j} \tag{3.4}$$

span the right kernel distribution of $\Lambda^{(0)}$, and this implies the validity of the algebraic conditions (2.5). Taking commutator

$$[V_i^{(0)}, V_j^{(0)}] = R^m{}_{kij} p_m \frac{\partial}{\partial p_k}, \tag{3.5}$$

we see that the distribution generated by $V_i^{(0)}$ is integrable iff the curvature tensor

$$R^m{}_{kij} = \frac{\partial \Gamma^m_{jk}}{\partial x^i} - \frac{\partial \Gamma^m_{ik}}{\partial x^j} + \Gamma^m_{in}\Gamma^n_{jk} - \Gamma^m_{jn}\Gamma^n_{ik} \tag{3.6}$$

of the metric g_{ij} vanishes. The left kernel distribution for $\Lambda^{(0)}$ is obtained by the complex conjugation of $\{V_i^{(0)}\}$ and therefore it is integrable whenever the right one is integrable. □

We see that the simple ansatz (3.1) for the metric on $T^*\mathcal{Q}$ induces a Kähler structure whenever the base Riemannian manifold $\mathcal{Q}$ is flat. If the curvature is nonzero we can try to modify the "bare" metric $G^{(0)}$ by adding higher powers in p's in order to restore the integrability of the kernel distributions. More precisely, we will allow the metric tensor on $T^*\mathcal{Q}$ to be given by formal power series in p's with smooth coefficients. The general expression for such a metric G looks like

$$G = G^{(0)} + G^{(1)} + G^{(2)} + G^{(3)} + \cdots, \tag{3.7}$$

where the components of the tensors $G^{(n)}$ are monomials in p's of degree n. Starting with $G^{(0)}$, the metric G will formally nondegenerate. We require the corresponding Hermitian form

$$\Lambda = G + \mathrm{i}\omega \tag{3.8}$$

to have an involutive kernel distribution $\{V_i\}$ of rank dim $\mathcal{Q}$. As the components of the metric tensor G, the components of null-vectors V_i are supposed to be given by formal powers series in p's, i.e.

$$V_i = V_i^{(0)} - i \sum_{n=1}^{\infty} X_{ki}{}^{j_1 \cdots j_n}(x) p_{j_1} \cdots p_{j_n} \frac{\partial}{\partial p_k} = V_i^{(0)} - i \sum_{n=1}^{\infty} X_{ki}^{(n)}, \tag{3.9}$$

so that

$$\Lambda(\cdot, V_i) = 0 \tag{3.10}$$

and

$$[V_i, V_j] = 0. \tag{3.11}$$

In fact, once the local vector fields V's satisfying (3.10) and (3.11) are known, the metric G is straightforwardly restored by them.

Proposition 2. *Given a commuting set of local vector fields* $\{V_i\}$ *of the form* (3.9) *so that the coefficients* $X_{ki}{}^{j_1\cdots j_n}$ *are tensors on* $\mathcal{Q}$ *being symmetric in ki and* $j_1\cdots j_n$. *Then the* V_i *span the right kernel distribution of the Hermitian form* $\Lambda = G + \mathrm{i}\omega$, *where*

$$G = \tilde{g}_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j + \tilde{g}^{ij}\tilde{D}p_i \otimes \tilde{D}p_j \tag{3.12}$$

and

$$\tilde{g}_{ij} = g_{ij} + \mathrm{Re}\sum_{n=1}^{\infty} X_{ij}^{(n)}, \qquad \tilde{D}p_i = \mathrm{d}p_i - \tilde{\Gamma}_{ji}^{k} p_k\,\mathrm{d}x^j,$$

$$\tilde{\Gamma}_{ij}^{k} p_k = \Gamma_{ij}^{k} p_k + \mathrm{Im}\sum_{n=1}^{\infty} X_{ij}^{(n)}, \tag{3.13}$$

$\tilde{g}^{ij}$ *being the formal inverse to the matrix* $\tilde{g}_{ij}$.

Proof. Using notation (3.13) one can write

$$V_i = \tilde{\nabla}_i - \mathrm{i}\tilde{g}_{ij}\frac{\partial}{\partial p_j}, \qquad \tilde{\nabla}_i = \frac{\partial}{\partial x^i} + \tilde{\Gamma}_{ij}^{k} p_k \frac{\partial}{\partial p_j}. \tag{3.14}$$

Then expressions (3.12), (3.14) for G and V_i formally coincide with those for $G^{(0)}$ and $V_i^{(0)}$. □

To show that Eq. (3.11) does have a solution satisfying the aforementioned conditions, we start with some preparatory constructions.

Consider the supercommutative algebra $\mathcal{F}$ of formal power series in p's with coefficients in the exterior algebra of differential forms on $\mathcal{Q}$. The general element of the algebra $\mathcal{F}$ reads

$$f(x, p, \mathrm{d}x) = \sum_{r,q=0}^{\infty} f_{k_1\cdots k_r}{}^{j_1\cdots j_q}(x) p_{j_1}\cdots p_{j_q}\,\mathrm{d}x^{k_1}\wedge\cdots\wedge \mathrm{d}x^{k_r}, \tag{3.15}$$

where the coefficients $f_{k_1\cdots k_r}^{j_1\cdots j_q}(x)$ make a tensor of type (q, r), symmetric in the contravariant indices and skew-symmetric in the covariant ones. Denote by $\mathcal{F}_{q,r} \subset \mathcal{F}$ the subspaces of homogenous elements.

Similarly, consider the Lie superalgebra $\mathcal{D}$ of first-order differential operators acting in $\mathcal{F}$ and having the form

$$Y(x, p, \mathrm{d}x) = \sum_{r,q=0}^{\infty}\Bigg(Y^m{}_{k_1\cdots k_r}{}^{j_1\cdots j_q}(x) p_{j_1}\cdots p_{j_q}\,\mathrm{d}x^{k_1}\wedge\cdots\wedge \mathrm{d}x^{k_r}\nabla_m$$
$$+ \tilde{Y}_{m\,k_1\cdots k_r}{}^{j_1\cdots j_q}(x) p_{j_1}\cdots p_{j_q}\,\mathrm{d}x^{k_1}\wedge\cdots\wedge \mathrm{d}x^{k_r}\frac{\partial}{\partial p_m}\Bigg), \tag{3.16}$$

where the coefficients $Y^m{}_{k_1\cdots k_r}{}^{j_1\cdots j_q}$ and $\tilde{Y}_{m\,k_1\cdots k_r}{}^{j_1\cdots j_q}$ make tensors of type $(q+1, r)$ and $(q, r+1)$, respectively, which are symmetric in $j_1\cdots j_q$ and skew-symmetric in $k_1\cdots k_r$. To each term of the series (3.16) we prescribe the bi-degree (q, r) and denote by $\mathcal{D}_{q,r} \subset \mathcal{D}$

the subspace of all such elements. The supercommutator of two homogeneous element from $\mathcal{D}$ is given by

$$[Y_1, Y_2]f = Y_1(Y_2 f) - (-1)^{r(Y_1)r(Y_2)} Y_2(Y_1 f), \quad \forall f \in \mathcal{F}.$$

Introduce the following two elements of the superalgebra $\mathcal{D}$:

$$\nabla = \mathrm{d}x^i \nabla_i, \qquad \delta = \mathrm{d}x^i g_{ij} \frac{\partial}{\partial p_j}, \qquad \nabla : \mathcal{F}_{q,r} \to \mathcal{F}_{q,r+1}, \qquad \delta : \mathcal{F}_{q,r} \to \mathcal{F}_{q-1,r+1}. \tag{3.17}$$

One can check that

$$\nabla^2 = \tfrac{1}{2}[\nabla, \nabla] = R, \qquad [\nabla, \delta] = 0, \qquad \delta^2 = \tfrac{1}{2}[\delta, \delta] = 0, \tag{3.18}$$

where

$$R = \frac{1}{2} R^m{}_{kij} p_m \, \mathrm{d}x^i \wedge \mathrm{d}x^j \frac{\partial}{\partial p_k}. \tag{3.19}$$

In what follows we will need the information about the cohomology of the nilpotent operator δ. This may be obtained by introducing a partial homotopy operator $\delta^{-1} : \mathcal{F}_{q,r} \to \mathcal{F}_{q+1,r-1}$ acting by the rule:

$$\delta^{-1} f = g^{ij} p_i \left(\frac{\partial}{\partial x^j} \right) \lrcorner \int_0^1 f(x, tp, t\,\mathrm{d}x) \frac{\mathrm{d}t}{t}, \tag{3.20}$$

where $Y \lrcorner f$ stands for the contraction of the vector field Y with the form f. Like δ, the operator δ^{-1} is nilpotent, $(\delta^{-1})^2 = 0$, and it satisfies the identity

$$\delta\delta^{-1} f + \delta^{-1}\delta f + \pi_0 f = f, \ \forall f \in \mathcal{F}, \tag{3.21}$$

where $\pi_0 : \mathcal{F} \to \mathcal{F}_{0,0} = C^\infty(\mathcal{Q})$ is the canonical projection, i.e. $\pi_0 f(x, p, \mathrm{d}x) = f(x, 0, 0)$. The last identity is similar to the Hodge–de Rham decomposition for the exterior algebra of differential forms and says that the space of δ-cohomology coincides with $\mathcal{F}_{0,0}$.

Note that the analogous Hodge–de Rham decomposition (3.21) takes place in the Lie superalgebra $\mathcal{D}$ if we put

$$\delta Y = [\delta, Y], \qquad \pi_0 Y(x, p, \mathrm{d}p) = Y(x, 0, 0), \quad \forall Y \in \mathcal{D},$$

and define δ^{-1} by the formula (3.20) in which $f(x, p, \mathrm{d}p)$ is substituted by $Y(x, p, \mathrm{d}p)$.

Consider differentiations $V \in \mathcal{D}_{\bullet,1}$ of the form

$$V = \mathrm{d}x^i V_i = \nabla - \mathrm{i}\delta - \mathrm{i}X, \tag{3.22}$$

where

$$X = \mathrm{d}x^i \sum_{n=1}^{\infty} X_{ki}{}^{j_1 \cdots j_n}(x) p_{j_1} \cdots p_{j_n} \frac{\partial}{\partial p_k} = \sum_{n=1}^{\infty} X^{(n)}. \tag{3.23}$$

Clearly, the involution condition (3.11) is equivalent to

$$V^2 = \tfrac{1}{2}[V, V] = 0. \tag{3.24}$$

In view of the obvious identities $\delta\theta = \nabla\theta = 0$, the symmetry condition $X_{ki}{}^{j_1\cdots j_n} = X_{ik}{}^{j_1\cdots j_n}$, $n = 1, 2, \ldots$, can be expressed as

$$V\theta = 0, \quad \theta = p_i\,\mathrm{d}x^i. \tag{3.25}$$

Thus, to construct the Kähler metric G of the form (3.7), it suffices to find V_i of the form (3.22) obeying Eqs. (3.24) and (3.25).

Proposition 3. *The general solution to* Eq. (3.24) *is given by the following recurrent formulae*:

$$X^{(1)} = \delta a^{(2)}, \qquad X^{(2)} = \delta^{-1}R - \delta^{-1}(\mathrm{i}\nabla X^{(1)} + \tfrac{1}{2}[X^{(1)}, X^{(1)}]) + \delta a^{(3)},$$

$$X^{(n)} = -\delta^{-1}\left(\mathrm{i}\nabla X^{(n-1)} + \frac{1}{2}\sum_{p=1}^{n-1}[X^{(p)}, X^{(n-p)}]\right) + \delta a^{(n+1)}, \quad n > 2, \tag{3.26}$$

where

$$a = \delta^{-1}X = \sum_{n=2}^{\infty}\frac{1}{n}a_k{}^{j_1\cdots j_n}(x)p_{j_1}\cdots p_{j_n}\frac{\partial}{\partial p_k} = \sum_{n=2}^{\infty}a^{(n)} \tag{3.27}$$

is an element from $\mathcal{D}_{\bullet,0}$.

Proof. Using Rels. (3.18) one finds

$$\tfrac{1}{2}(V)^2 = R - \mathrm{i}\nabla X - \delta X - \tfrac{1}{2}X^2, \tag{3.28}$$

where we have put $\nabla X \equiv [\nabla, X]$. Eq. (3.24) can be written now as

$$\delta X = R - \mathrm{i}\nabla X - \tfrac{1}{2}X^2. \tag{3.29}$$

Applying δ^{-1} to both sides of the last equation and accounting the Hodge–de Rham decomposition (3.21) for the Lie superalgebra $\mathcal{D}$, we get

$$X = \delta^{-1}R - \delta^{-1}(\mathrm{i}\nabla X + \tfrac{1}{2}X^2) + \delta a. \tag{3.30}$$

Since the operator δ^{-1} increases the degree of p's, while ∇ remains it intact, Eq. (3.30) can be solved by iterations and Rels. (3.26) represent the corresponding recurrent formulae.

It remains to show that any solution $\tilde{X}$ to Eq. (3.30) obeys the initial Eq. (3.29) and the condition (3.27). First of all, the relation $\delta^{-1}\tilde{X} = a$ immediately follows from (3.30) as consequence of the nilpotency of the operator δ^{-1} and the Hodge–de Rham decomposition for a. Denote $\tilde{V} = \nabla - \mathrm{i}\delta - \mathrm{i}\tilde{X}$. Then

$$\tfrac{1}{2}\tilde{V}^2 = R - \mathrm{i}\nabla\tilde{X} - \delta\tilde{X} - \tfrac{1}{2}\tilde{X}^2 \tag{3.31}$$

and $\delta^{-1}(\tilde{V}^2) = 0$. On the other hand, writing the Jacobi identity $(\tilde{V})^3 = (1/4)[\tilde{V}, [\tilde{V}, \tilde{V}]] = 0$ in the form

$$\delta(\tilde{V}^2) = (-\mathrm{i}\tilde{V} + \delta)\tilde{V}^2 = (-\mathrm{i}\nabla - \tilde{X})\tilde{V}^2$$

and using the Hodge–de Rham decomposition together with $\delta^{-1}(\tilde{V}^2) = 0$, we get

$$\tilde{V}^2 = -\delta^{-1}((\mathrm{i}\nabla + \tilde{X})\tilde{V}^2). \tag{3.32}$$

Since the operator $(\mathrm{i}\nabla + \tilde{X})$ does not decrease the degree of p's, while δ^{-1} increases the degree by one, we conclude that Eq. (3.32) has the unique solution $\tilde{V}^2 = 0$. □

To formulate the next proposition introduce the Liouville vector field on $T^*\mathcal{Q}$:

$$\mathcal{D}_{1,0} \ni \hat{N} = p_i \frac{\partial}{\partial p_i}. \tag{3.33}$$

Proposition 4. *In order for V_i defined by Eq. (3.26) to satisfy the condition (3.25) it is necessary and sufficient that the element*

$$\mathcal{F}_{\bullet,1} \ni \tilde{a} = (\hat{N}a)\theta = \sum_{n=2}^{\infty} \mathrm{d}x^k a_k{}^{j_1\cdots j_n}(x)p_{j_1}\cdots p_{j_n} = \sum_{n=2}^{\infty} \tilde{a}^{(n)} \tag{3.34}$$

determined by Eq. (3.27) obeys the equation

$$V\tilde{a} = 0. \tag{3.35}$$

The general solution to the last equation is given by the following recurrent relations:

$$\tilde{a}^{(2)} = \delta b^{(3)}, \qquad \tilde{a}^{(n)} = -\delta^{-1}\left(\mathrm{i}\nabla\tilde{a}^{(n-1)} + \sum_{p=1}^{n-2} X^{(p)}\tilde{a}^{(n-p)}\right) + \delta b^{(n+1)}, \quad n \geq 3, \tag{3.36}$$

where

$$b = \delta^{-1}\tilde{a} = \sum_{n=3}^{\infty} \frac{1}{n} b^{j_1\cdots j_n}(x)p_{j_1}\cdots p_{j_n} = \sum_{n=3}^{\infty} b^{(n)} \tag{3.37}$$

is an arbitrary element from $\mathcal{F}_{\bullet,0}$ whose expansion starts with third order in p's.

Proof. We should analyze the condition

$$X\theta = 0 \Leftrightarrow X^{(n)}\theta = 0. \tag{3.38}$$

Clearly, $X^{(1)}\theta = 0$ implies $\delta\tilde{a}^{(2)} = 0$, and this is equivalent to the validity of Eq. (3.35) in first order in p's. The equivalence between Eqs. (3.35) and (3.25) can be now established by simple induction in n.

Making use of the formulae (3.26) and the Bianchi identity $(\delta^{-1}R)\theta = 0$, one can rewrite

$$X^{(n)}\theta = -\delta^{-1}\left(\mathrm{i}\nabla X^{(n-1)} + \frac{1}{2}\sum_{p=1}^{n-1}[X^{(p)}, X^{(n-p)}]\right)\theta + \delta(a^{(n+1)}\theta) = 0, \quad n > 1, \tag{3.39}$$

or more explicitly

$$X^{(n)}\theta = -\frac{1}{n+1}\,\mathrm{d}x^m \wedge \mathrm{d}x^k \Bigg[i\nabla^{j_1} X_{km}{}^{j_2\cdots j_n} - i\nabla_m X_k{}^{j_1 j_2\cdots j_n} + \sum_{p=1}^{n-1}(n-p) \times (X_s{}^{j_1 j_2\cdots j_{p+1}} X_{km}{}^{s j_{p+2}\cdots j_n} - X_{sm}{}^{j_1\cdots j_p} X_k{}^{j_{p+1} s j_{p+2}\cdots j_n}) \Bigg] \times p_{j_1}\cdots p_{j_n} + \delta(a^{(n+1)}\theta). \tag{3.40}$$

Here all indices are risen and lowered with the help of g^{ij} and g_{ij}.

Proceeding by induction, we suppose that all the equations $X^{(l)}\theta = 0$ with $l < n$ are already satisfied, i.e. $X^{(l)}_{km} = X^{(l)}_{mk}$. Then Eq. (3.40) takes the form

$$X^{(n)}\theta = \frac{1}{n+1}\,\mathrm{d}x^m \wedge \mathrm{d}x^k \Bigg[i\nabla_m X_k{}^{j_1 j_2\cdots j_n} + \sum_{p=1}^{n-1}(n-p) X_{sm}{}^{j_1\cdots j_p} X_k{}^{j_{p+1} s j_{p+2}\cdots j_n} \Bigg] \times p_{j_1}\cdots p_{j_n} + \delta(a^{(n+1)}\theta). \tag{3.41}$$

Notice that Eq. (3.27) implies that

$$X_k{}^{j_1\cdots j_n} p_{j_1}\cdots p_{j_n} = a_k{}^{j_1\cdots j_n} p_{j_1}\cdots p_{j_n}. \tag{3.42}$$

Taking in account symmetry of X and a in contravariant indices one can bring Eq. (3.42) into the form

$$(n-p) X_k{}^{j_{p+1} s j_{p+2}\cdots j_n} + X_k{}^{s j_{p+1} j_{p+2}\cdots j_n} = (n-p+1) a_k{}^{s j_{p+1} j_{p+2}\cdots j_n}. \tag{3.43}$$

Substituting Eqs. (3.42) and (3.43) into (3.41) and using the identity

$$\mathrm{d}x^m \wedge \mathrm{d}x^k \sum_{p=1}^{n-1} X_{sm}{}^{j_1\cdots j_p} X_k{}^{s j_{p+1} j_{p+2}\cdots j_n} p_{j_1}\cdots p_{j_n} = 0,$$

we get

$$X^{(n)}\theta = \frac{1}{n+1}\,\mathrm{d}x^m \wedge \mathrm{d}x^k \Bigg[i\nabla_m a_k{}^{j_1\cdots j_n} + \sum_{p=1}^{n-1}(n-p+1) X_{sm}{}^{j_1\cdots j_p} a_k^{s j_{p+1} j_{p+2}\cdots j_n} \Bigg] \times p_{j_1}\cdots p_{j_n} + \delta(a^{(n+1)}\theta).$$

Finally, comparing the last expression with definition (3.34) we find

$$X^{(n)}\theta = \frac{\mathrm{i}}{n+1}\Bigg[\nabla\tilde{a}^{(n)} - \mathrm{i}\delta\tilde{a}^{(n+1)} - \mathrm{i}\sum_{p=1}^{n-1} X^{(p)}\tilde{a}^{(n-p+1)} \Bigg] = \frac{\mathrm{i}}{n+1}(V\tilde{a})^{(n)}. \tag{3.44}$$

Returning to Eq. (3.39) we see, that $X^{(n)}\theta = 0$ is equivalent to $(V\tilde{a})^{(n)} = 0$, provided all the equations $X^{(l)}\theta = 0$ with $l < n$ are satisfied, but this implies (3.35). □

Thus we have shown that the ambiguity in the recurrent definition (3.26) of the Kähler structure is completely described by one function $b(x, p)$ of the form (3.37). Having a set of formal functions $X_{ij}(x, p)$, symmetric in indices ij, one can immediately reconstruct the Kähler metric G by the formula (3.12).

It is also possible to rewrite Λ in a form which makes explicit the existence of the right/left kernel distributions. By definition, the local vector fields (3.9) generate the Kähler polarization of the cotangent bundle, in particular,

$$\omega(V_i, V_j) = 0. \tag{3.45}$$

Alternatively, one can say that the one-forms

$$Z_i = -V_i \lrcorner \omega = \tilde{D} p_i + \mathrm{i}\tilde{g}_{ik}\,\mathrm{d}x^k = \mathrm{d}p_i - \Gamma^k_{ji} p_k\,\mathrm{d}x^j + \mathrm{i}g_{ij}\,\mathrm{d}x^j + \mathrm{i}X_{ji}(x, p)\,\mathrm{d}x^j \tag{3.46}$$

generate the annihilator of the holomorphic distribution $\{V_i\}$. By the Frobenius theorem, the annihilator is closed under the exterior differentiation. A straightforward computation yields

$$\mathrm{d}Z_i = \left(-\Gamma^j_{ik} + \mathrm{i}\frac{\partial X_{ik}}{\partial p_j}\right) Z_j \wedge \mathrm{d}x^k. \tag{3.47}$$

In terms of the basis one-forms $(Z_i, \overline{Z}_j)$, the Hermitian form Λ can be written as

$$\Lambda = \tilde{g}^{ij}\,\overline{Z}_i \otimes Z_j. \tag{3.48}$$

In view of (3.45) $V_i \lrcorner Z_j = 0$, and thus $\Lambda(\cdot, V_i) = \Lambda(\overline{V}_i, \cdot) = 0$.

In [10] it was shown that the Hermitian form Λ contains all the geometric prerequisites needed for the Wick symbol construction (see also [11]).

4. Holomorphic coordinates and convergence of the power series for the metric

Since the vector fields $\{V_i\}$ (3.9) give a basis in the holomorphic distribution, we arrive at the following definition of anti-holomorphic functions on $T^*\mathcal{Q}$. The formal series

$$f = \sum_{n=0}^{\infty} f^{j_1\cdots j_n}(x) p_{j_1}\cdots p_{j_n} \equiv \sum_{n=0}^{\infty} f^{(n)} \in \mathcal{F}_{\bullet,0} \tag{4.1}$$

is said to be a formal anti-holomorphic function if

$$Vf = 0. \tag{4.2}$$

Proposition 5. *Let f be a formal anti-holomorphic function, then*

$$\mathrm{d}f = \frac{\partial f}{\partial p_i} Z_i \tag{4.3}$$

where one forms Z_i (3.46) *span the annihilator of the holomorphic distribution* $\{V_i\}$.

Proof. This immediately follows from the identity

$$\mathrm{d}f = \frac{\partial f}{\partial p_i} Z_i + Vf, \quad V = \mathrm{d}x^i V_i,$$

f being an arbitrary formal function on $T^*\mathcal{Q}$. □

Eq. (4.2) can be solved iteratively in full analogy with Eqs. (3.24) and (3.35). At the nth step the solution reads

$$f^{(n)} = -\mathrm{i}\delta^{-1}\left(\nabla f^{(n-1)} - \mathrm{i}\sum_{r=1}^{n-1} X^{(r)} f^{(n-r)}\right), \quad n \geq 1, \tag{4.4}$$

where $f^{(0)}(x) = f|_{p=0}$ is a given function in the coordinate chart $(U, \{x^i\})$ and δ^{-1} is the partial homotopy operator (3.20). If $f^{(0)n}(x), n = 1, \ldots, \dim \mathcal{Q}$, is a set of independent functions on U the corresponding anti-holomorphic functions $f^n(x, p)$ can be taken as anti-holomorphic coordinates adapted to the formal Kähler structure on $T^*\mathcal{Q}$.

So far we have dealt with the construction of formal Kähler structure on $T^*\mathcal{Q}$ assuming all the ingredients are given by formal power series in the canonical fiber coordinates. Now, we are going to reproduce this construction from a somewhat different, perhaps more geometrical, viewpoint. Namely, assuming $(\mathcal{Q}, g)$ to be real-analytical Riemannian manifold, we define an atlas of local holomorphic coordinates in a tubular neighborhood $W \subset T^*\mathcal{Q}$ of the base $\mathcal{Q}$. The coordinates are constructed as solutions of a differential equation associated to a certain Hamiltonian flow on $T^*\mathcal{Q}$ with initial data on $\mathcal{Q}$. The complex structure, corresponding to these coordinates, turns out to be compatible with the canonical symplectic form and thus it defines an integrable Kähler structure. The latter is shown to coincide with the formal Kähler structure of Section 3, that implies the convergence of the power series (3.7) in W.

The main idea behind this construction is very simple. Suppose that X satisfies the special boundary condition $\delta^{-1}X = 0$ (see Eq. (3.27)). Then the second term in the l.h.s. of Eq. (4.4) vanishes and we have

$$f^{(n)} = -\mathrm{i}\delta^{-1}\nabla f^{(n-1)} = -\frac{\mathrm{i}}{n} g^{ij} p_i \nabla_j f^{(n-1)} = \frac{1}{n}\{H_0, f^{(n-1)}\}, \quad n \geq 1, \tag{4.5}$$

where

$$H_0 = \frac{\mathrm{i}}{2} g^{ij}(x) p_i p_j. \tag{4.6}$$

The formula (4.5) implies that $f(x, p) = f(x, p; 1)$, where $f(x, p, t)$ is the solution of the equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H_0, f\} \tag{4.7}$$

with the initial condition $f(x, p; 0) = f^{(0)}(x)$. We see that the series defined by the recurrence relation (4.5) is nothing but the exponential map generated by the complex geodesic Hamiltonian H_0.

Now let us choose a set $\{f^{(0)n}, n = 1, \ldots, \dim \mathcal{Q}\}$ of independent functions on U, say $f^{(0)n} = x^n$, and let $f^n(x, p) = f^n(x, p; 1)$ be their exponential lift on $T^*\mathcal{Q}$ w.r.t. the Hamiltonian (4.6). The standard theorems of ODE's ensure the real-analyticity of functions $f^n(x, p)$ in a sufficiently small neighborhood $W \cap T^*U$.

By construction, the complex functions $f^n(x, p)$ pair-wise commute w.r.t. the canonical Poisson bracket,

$$\{f^m, f^n\} = 0. \tag{4.8}$$

The Kähler metric is now defined as the matrix inverse of the Poisson brackets of the holomorphic and anti-holomorphic coordinates

$$G^{m\bar{n}} = \frac{1}{2}\{f^m, \bar{f}^n\} \equiv \frac{1}{2}\left(\frac{\partial f^m}{\partial x^k}\frac{\partial \bar{f}^n}{\partial p_k} - \frac{\partial \bar{f}^n}{\partial x^k}\frac{\partial f^m}{\partial p_k}\right). \tag{4.9}$$

Clearly, $(G^{n\bar{m}})$ is a tensor w.r.t. coordinate changes on W.

This Kähler structure actually coincides with the restriction on W of the formal Kähler structure constructed in the previous section. In order to see this, we express the Hermitian form Λ (see Eq. (3.48)) in terms of the holomorphic coordinates f^n.

Proposition 6. *With the above definition we have*

(i) *The matrix*

$$e^{mi} = \frac{\partial f^m}{\partial p_i}, \quad m, i = 1, \ldots, \dim \mathcal{Q} \tag{4.10}$$

is invertible for W being sufficiently small.

(ii) *The one-forms*

$$Z_i = e_{im}\, \mathrm{d} f^m, \qquad e^{mi}e_{ik} = \delta^m_k \tag{4.11}$$

are given by Eq. (3.46), *where $X_{ij}(x, p)$ are real-analytical functions satisfying the conditions*

$$X_{ij}(x, p) = X_{ji}(x, p), \qquad X_{ij}(x, 0) = 0, \qquad p_k g^{ki} X_{ij}(x, p) = 0. \tag{4.12}$$

(iii) *The real-analytical vector fields V_i, defined by the equation $V_i \lrcorner\, \omega = -Z_i$, have the form* (3.14) *with*

$$\tilde{g}_{ij} = \tilde{g}_{ji} = \mathrm{Im}\left(e_{jm}\frac{\partial f^m}{\partial x^i}\right) = G^{k\bar{m}} e_{ki}\bar{e}_{mj} = g_{ij} + O(p^2),$$

$$\tilde{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ji} = -\mathrm{Re}\left(e_{jm}\frac{\partial f^m}{\partial x^i}\right) = \Gamma^k_{ij} + O(p). \tag{4.13}$$

(iv) *The vector fields V_i commute pair-wise and generate a local basis of the holomorphic distribution, so that*

$$V_i f^n = 0. \tag{4.14}$$

(v) *The Hermitian form Λ is given by* Eq. (3.48), *where Z_j and $\tilde{g}^{ij}$ are defined by* (ii) *and* (iii).

Given holomorphic coordinates f^n, this proposition allows to reconstruct the real-analytical V_i and Λ subject to the same defining conditions which have been imposed on their formal counterparts from Section 3 with $\delta^{-1}X = 0$.

Proof. By the Cauchy theorem the functions f^n are real-analytical as solutions of the differential equation (4.7) with a real-analytical r.h.s. and the initial conditions $f^{(0)n} = x^n$. In lower orders in p_i we have

$$f^n = x^n - \mathrm{i}g^{nj}p_j - \tfrac{1}{2}\Gamma^n_{kl}g^{ki}g^{lj}p_ip_j + O(p^3). \tag{4.15}$$

and

$$e^{ni} = \frac{\partial f^n}{\partial p_i} = -\mathrm{i}g^{ki}(\delta^n_k + \mathrm{i}\Gamma^n_{kl}g^{lj}p_j) + O(p^2). \tag{4.16}$$

So, the matrix (e^{ni}) is non-degenerate for sufficiently small p_i and the inverse matrix reads

$$e_{in} = \mathrm{i}g_{ik}(\delta^k_n - \mathrm{i}\Gamma^k_{nl}g^{lj}p_j) + O(p^2). \tag{4.17}$$

From the definition (4.11), we get

$$Z_i = e_{im}\left(\frac{\partial f^m}{\partial p_j}\,\mathrm{d}p_j + \frac{\partial f^m}{\partial x^j}\,\mathrm{d}x^j\right) = \mathrm{d}p_i + e_{im}\frac{\partial f^m}{\partial x^j}\,\mathrm{d}x^j. \tag{4.18}$$

Using (4.15) and (4.17) one can write

$$e_{im}\frac{\partial f^m}{\partial x^j} = -\Gamma^k_{ij}p_k + \mathrm{i}g_{ij} + \mathrm{i}X_{ij}(x,p), \quad X_{ij}(x,p) = O(p^2). \tag{4.19}$$

Furthermore, the involution conditions (4.8) can be brought in the form

$$e_{im}\frac{\partial f^m}{\partial x^j} = e_{jm}\frac{\partial f^m}{\partial x^i} \tag{4.20}$$

that implies the symmetry of X_{ij}, $\tilde{g}_{ij}$ and $\tilde{\Gamma}^k_{ij}$ in ij.

From the definition $V_i \lrcorner\, \omega = -Z_i$ we find

$$V_i = \frac{\partial}{\partial x_i} - e_{jm}\frac{\partial f^m}{\partial x^i}\frac{\partial}{\partial p_j} \tag{4.21}$$

and thus $V_if^m = e_{ik}\{f^m, f^k\} = 0$. A direct computation shows that $[V_i, V_j] = 0$, so (iv) is proven.

The Kähler metric G can be straightforwardly rewritten in terms of Z_i

$$G \equiv G_{\bar{n}m}\,\mathrm{d}\overline{f^n}\,\mathrm{d}f^m = \tilde{g}^{ij}\overline{Z}_iZ_j, \quad \tilde{g}^{ij} = G_{\bar{n}m}\overline{e^{ni}}e^{mj}. \tag{4.22}$$

The inverse matrix to $(\tilde{g}^{ij})$ reads

$$\tilde{g}_{ij} = G^{n\bar{m}}e_{in}\overline{e}_{jm} = \frac{1}{2}\left(e_{im}\frac{\partial f^m}{\partial x^j} - \overline{e}_{im}\frac{\partial \overline{f^m}}{\partial x^j}\right) = \operatorname{Im} e_{im}\frac{\partial f^m}{\partial x^j} \tag{4.23}$$

and this implies (v).

It only remains to prove that $X_{ij}g^{jk}p_k = 0$ or, what is the same, $\delta^{-1}X = 0$. A solution to Eq. (4.7) is nothing but the geodesic of the complex metric $\mathrm{i}g_{ij}(x)$, which pass, at $t = 0$, through the point $\{x^i\} \in \mathcal{Q}$ with the tangent vector $\{g^{ij}p_j\} \in T_x\mathcal{Q}$. Accounting for the homogeneity property of solutions of the geodesic equation, $f^n(x, \lambda p, \lambda^{-1} t) = f^n(x, p, t)$, $\forall\lambda \in \mathbb{R} \setminus \{0\}$, one can write

$$\frac{\mathrm{d} f^n}{\mathrm{d} t} = p_i \frac{\partial f^n}{\partial p_i} = \{H_0,\ f^n\}, \tag{4.24}$$

that yields

$$g^{ik} p_k e_{jn} \frac{\partial f^n}{\partial x^i} = \mathrm{i} g^{ik} p_k (g_{ij} - \Gamma^l_{ij} p_l). \tag{4.25}$$

Comparing the last equation with (4.19) we conclude that $X_{ij} g^{jk} p_k = 0$. □

In the above consideration, we have used the complex geodesic Hamiltonian (4.6) to define an integrable complex structure on W. In fact, this construction works for an arbitrary analytical Hamiltonian. In particular, one can consider a Hamiltonian of the form

$$H = H_0 + \Delta H, \quad \Delta H = \sum_{n=3}^{\infty} h^{j_1 \cdots j_n}(x) p_{j_1} \cdots p_{j_n} \tag{4.26}$$

and define an atlas of holomorphic coordinates in W using the exponential map generated by the Hamiltonian flow. One can see that the resulting Kähler structure corresponds to that from Section 3 with nonvanishing boundary condition $\delta^{-1} X \neq 0$. The function $\Delta H(x, p)$ can be thought of as a function corresponding to the ambiguity of the formal Kähler structure described by the function $b(x, p)$ (3.37).

5. The first Chern class

In the previous sections $T^*\mathcal{Q}$ has been equipped with a formal Kähler structure Λ (3.48). Assuming $(\mathcal{Q}, g)$ to be real-analytical, we have also shown, that the formal series for the lifted Kähler metric G converges in a neighborhood of the zero section in $T^*\mathcal{Q}$. In the context of deformation quantization, it is important to identify the cohomology class of the Ricci form ϱ. It is shown in [10], that the Wick- and Weyl-type quantizations on a finite-dimensional Kähler manifold are equivalent to each other iff the corresponding Ricci form is exact, i.e. the corresponding cohomology class $[\varrho]$ equals to zero. For arbitrary Kähler manifold $[\varrho]$ is known to depend only on the complex structure and it is proportional to the first Chern class of the manifold.

Proposition 7. *The Ricci form of the Kähler structure Λ is exact.*

Proof. Because $T^*\mathcal{Q}$ (as well as any tubular neighborhood of zero section) is clearly homotopic to the base, the canonical embedding $i : \mathcal{Q} \to T^*\mathcal{Q}$ induces the isomorphism of de Rham cohomology $H(T^*\mathcal{Q}) \to H(\mathcal{Q})$. In particular, the Ricci form ϱ of a Kähler structure on T^*Q is exact iff the same is the pull-back form $i^*\varrho$ on $\mathcal{Q}$.

Recall the definition of the Ricci form. Let $\{x^a\}$ be a set of local coordinates on a Kähler manifold, then

$$\varrho = \omega^{ab} K_{abcd}\, \mathrm{d}x^c \wedge \mathrm{d}x^d, \tag{5.1}$$

where ω^{ab} and K_{abcd} are the components of the Poisson and Riemannian tensors associated to the Kähler two-form and metric, respectively.

In Section 3 the metric G is defined by the formal series (3.13) in the fiber coordinates, so, the Riemannian tensor K and Ricci form ϱ are formal series as well. By definition, $i^*\varrho$ is determined by that term of the series ϱ, which does not depend on p_i and $\mathrm{d}p_i$. A direct calculation of this term gives

$$i^*\varrho = \mathrm{d}\psi, \quad \psi = 2g_{il}g_{jk}\,\mathrm{Re}(b^{ljk})\,\mathrm{d}x^i, \tag{5.2}$$

where b^{ljk} is the lower order term of the series (3.37). Hence, one can see that $i^*\varrho$ is exact, and therefore $[\rho] = 0$. □

The first Chern class of the Kähler structure on $T^*\mathcal{Q}$ thus turns out to be zero. As mentioned above, this implies the equivalence between the Wick and Weyl deformation quantizations for systems with finite number degrees of freedom. Generally speaking, this reason cannot be extended to the field theory and we turn to the discussion on this problem in Section 7.

6. Examples

In this section we illustrate the general construction by several examples with special emphasis on a possible application to quantum field theory.

6.1. *Cotangent bundle of a constant curvature space*

In this case, all the covariants of the metric g are expressed algebraically via the metric itself. In particular, the covariant curvature tensor is given by

$$R_{mkij} = K(g_{mi}g_{kj} - g_{mj}g_{ki}), \quad K = \mathrm{const} \tag{6.1}$$

(positive curvature corresponds to $K > 0$). The metrics of constant curvature are known also as those admitting the maximal number of isometries. The action of the isometry group on $\mathcal{Q}$ is naturally lifted to the transformation group of the cotangent bundle $T^*\mathcal{Q}$. The orbits of the latter group are given by the level sets $h = \mathrm{const}$ of the geodesic Hamiltonian $h = g^{ij}(x)p_ip_j$ which thus generates the whole ring of invariant functions on $T^*\mathcal{Q}$.

Consider the special class of Kähler metrics on $T^*\mathcal{Q}$ which are invariant under the action of the isometry group. In this case the formal function $b(x, p)$, entering to the general solution for the Kähler metric, should be a function of the geodesic Hamiltonian alone, i.e. $b(x, p) = f(h)$, f being an analytical function vanishing at zero. Then the local vector fields V_i, being constructed by the formulae (3.26), will clearly have the form

$$V_i = \nabla_i - \mathrm{i}(A(h)g_{im} + B(h)p_ip_m)\frac{\partial}{\partial p_m}, \tag{6.2}$$

where $A(h)$, $B(h)$ are complex-valued functions of h, $A(0) = 1$. Taking commutators, we find

$$[V_i, V_j] = (2A'(A + hB) - AB - K)(g_{im}p_j - g_{jm}p_i)\frac{\partial}{\partial p_m}, \tag{6.3}$$

where the prime denotes the derivative in h. Thus, the local vector fields V_i are in involution iff

$$2A'(A + hB) - AB - K = 0. \tag{6.4}$$

For a fixed $B(h)$, this gives us a first-order differential equation on $A(h)$ with initial condition $A(0) = 1$. In fact, the function $B(h)$ is uniquely determined by $b(x, p) = f(h)$. The recurrent procedure (3.26) can then been understood as finding a solution for the differential equation (6.4) in terms of power series in the momenta p_i. Given the functions $A(h)$ and $B(h)$ satisfying Eq. (6.4), the Kähler metric G on $T^*\mathcal{Q}$ is defined by Eq. (3.12) with

$$\begin{aligned}\tilde{g}_{ij} &= \mathrm{Re}(A(h)g_{ij} + B(h)p_i p_j),\\ \tilde{D}p_i &= \mathrm{d}p_i - \Gamma^k_{ji} p_k\,\mathrm{d}x^j - \mathrm{Im}(A(h)g_{ij} + B(h)p_i p_j)\,\mathrm{d}x^j.\end{aligned} \tag{6.5}$$

Choosing, for simplicity, the functions $A(h)$ and $B(h)$ to be real-valued, we find

$$\begin{aligned}G &= (A(h)g_{ij} + B(h)p_i p_j)\,\mathrm{d}x^i \otimes \mathrm{d}x^j\\ &\quad + \frac{1}{A(h)}\left(g^{ij} - \frac{B(h)g^{im}g^{jn}p_m p_n}{A(h) + hB(h)}\right) Dp_i \otimes Dp_j.\end{aligned} \tag{6.6}$$

The general theorems for the ordinary differential equations ensure the non-degeneracy and smoothness of G in a sufficiently small tubular neighborhood of $\mathcal{Q}$ in $T^*\mathcal{Q}$ (if one identifies $\mathcal{Q}$ with the zero section of the cotangent bundle). As is seen, the singularities of the metric (6.6) may appear as the points at which either $A(h) = 0$ or $A(h) + hB(h) = 0$. From the viewpoint of the recurrent procedure, the presence of singular points implies a finite radius of convergence for the power series in p's. To illustrate this situation, consider three special cases when Eq. (6.4) is explicitly integrable.

(i) $B = 0$, $A = \sqrt{1 + Kh}$,

$$G = \sqrt{1 + Kh}\, g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j + \frac{g^{ij} Dp_i \otimes Dp_j}{\sqrt{1 + Kh}}. \tag{6.7}$$

When $K > 0$, the Kähler metric is well-defined on the whole $T^*\mathcal{Q}$, whereas for $K < 0$, the singularities of the metric form the surface $h = -K^{-1}$. Clearly, in both cases the radius of convergence of the power series in p's equals $|K|^{-1}$.

(ii) $B = -K$, $A = 1$,

$$G = (g_{ij} - Kp_i p_j)\,\mathrm{d}x^i \otimes \mathrm{d}x^j + \left(g^{ij} + \frac{Kg^{im}g^{jn}p_m p_n}{1 - Kh}\right) Dp_i \otimes Dp_j. \tag{6.8}$$

The situation is opposite to the previous one: the Kähler metric is smooth over the whole manifold $T^*\mathcal{Q}$ when $K < 0$, while for $K > 0$ the singular surface appears as the level set of the geodesics Hamiltonian $h = K^{-1}$.

(iii) One more interesting case when Eq. (6.4) is integrable in elementary functions corresponds to the choice $A + hB = 1$. In the language of the recurrent procedure this is equivalent to the condition $\delta^{-1}X = 0$. When $K < 0$ we find

$$A = \sqrt{|K|h}\,\mathrm{ctg}\,\sqrt{|K|h}$$

and

$$G = \frac{\sqrt{|K|h}}{\mathrm{tg}\,\sqrt{|K|h}}\left(g_{ij} + \frac{1}{h}\left(\frac{\mathrm{tg}\,\sqrt{|K|h}}{\sqrt{|K|h}} - 1\right)p_i p_j\right)\mathrm{d}x^i \otimes \mathrm{d}x^j$$
$$+ \frac{\mathrm{tg}\,\sqrt{|K|h}}{\sqrt{|K|h}}\left(g^{ij} + \frac{1}{h}\left(\frac{\sqrt{|K|h}}{\mathrm{tg}\,\sqrt{|K|h}} - 1\right)g^{im}g^{jn}p_m p_n\right)Dp_i \otimes Dp_j. \quad (6.9)$$

The singularities of the metric form the surface $h = \pi^2(4|K|)^{-1}$. For $K > 0$, the solution can be obtained from (6.9) by simple replacement $\mathrm{tg}\,\sqrt{|K|h} \to \mathrm{th}\,\sqrt{Kh}$. Clearly, the resulting metric appears to be well-defined on the whole cotangent bundle $T^*\mathcal{Q}$.

6.2. *Harmonic oscillator and free scalar field*

This is the fundamental physical model in context of which the notion of Wick quantization has first appeared. In this case, the Kähler structure is usually attributed to the representation of creation-annihilation operators resulting from the canonical quantization of the oscillatory variables

$$a_\alpha = \frac{1}{\sqrt{2\omega_\alpha}}(P_\alpha - \mathrm{i}\omega_\alpha Q_\alpha), \qquad \bar{a}_\alpha = \frac{1}{\sqrt{2\omega_\alpha}}(P_\alpha + \mathrm{i}\omega_\alpha Q_\alpha),$$
$$\{a_\alpha, \bar{a}_\beta\} = -\mathrm{i}\delta_{\alpha\beta}, \qquad \{a_\alpha, a_\beta\} = \{\bar{a}_\alpha, \bar{a}_\beta\} = 0, \quad \alpha, \beta = 1, \ldots, n. \quad (6.10)$$

Here Q_α, P_α are normal coordinates and momenta associated to the normal frequencies ω_α of an n-dimensional harmonic oscillator.

From the geometrical viewpoint, a_α, $\bar{a}_\alpha$ are nothing but the holomorphic and anti-holomorphic coordinates adapted to the flat Kähler metric

$$G = \sum_{\alpha=1}^{n}(\mathrm{d}\bar{a}_\alpha \otimes \mathrm{d}a_\alpha + \mathrm{d}a_\alpha \otimes \mathrm{d}\bar{a}_\alpha) = \sum_{\alpha=1}^{n}\left(\frac{1}{\omega_\alpha}\,\mathrm{d}P_\alpha \otimes \mathrm{d}P_\alpha + \omega_\alpha \mathrm{d}Q_\alpha \otimes \mathrm{d}Q_\alpha\right). \quad (6.11)$$

It is instructive to rewrite this metric in arbitrary linear canonical coordinates x^i, p_i. Denote by $M = (M_{ij})$, $K = (K_{ij})$ the corresponding matrices of mass and stiffness, then the Hamiltonian of the harmonic oscillator reads

$$H = \sum_{\alpha=1}^{n}\omega_\alpha \bar{a}_\alpha a_\alpha = \frac{1}{2}(M^{ij}p_i p_j + K_{ij}x^i x^j). \quad (6.12)$$

A simple linear algebra yields

$$G = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j + g^{ij}\,\mathrm{d}p_i \otimes \mathrm{d}p_j, \quad (6.13)$$

where $g = (g_{ij})$ is given by

$$g = M\sqrt{M^{-1}K}. \tag{6.14}$$

Recall that the matrix $M^{-1}K$ is diagonalizable and its eigenvalues coincide with the squares of normal frequencies. Thus, we see that the Kähler metric on the configuration space of the harmonic oscillator is given by expression (6.14). In particular, for the isotropic oscillator $K = \omega^2 M$ and $g = \omega M$.

This construction is straightforwardly generalized to the case of free fields, treated as a continual set of harmonic oscillators. The free Hamiltonian of one scalar field on the $(d+1)$-dimensional Minkowski space reads

$$H = \frac{1}{2}\int \mathrm{d}\mathbf{x}(\pi^2(\mathbf{x}) + \partial_\mu\phi(\mathbf{x})\partial_\mu\phi(\mathbf{x}) + m^2\phi^2(\mathbf{x})), \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \mu = 1, \dots, d. \tag{6.15}$$

The formal comparison with the Hamiltonian of the n-dimensional oscillator suggests the following identification for the mass and stiffness matrices:

$$M(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \qquad K(\mathbf{x}, \mathbf{y}) = (-\Delta_{\mathbf{x}} + m^2)\delta(\mathbf{x} - \mathbf{y}). \tag{6.16}$$

Correspondingly, the infinite dimensional counterparts to the matrix (6.14) and its inverse are given by

$$g(\mathbf{x}, \mathbf{y}) = \sqrt{-\Delta_{\mathbf{x}} + m^2}\,\delta(\mathbf{x} - \mathbf{y}) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^d}\omega(\mathbf{p})\,\mathrm{e}^{\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{y})},$$

$$g^{-1}(\mathbf{x}, \mathbf{y}) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^d}\frac{1}{\omega(\mathbf{p})}\,\mathrm{e}^{\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{y})}, \quad \omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}. \tag{6.17}$$

The value of the Kähler metric (3.12) on the tangent field $\Phi = (\delta\phi(\mathbf{x}), \delta\pi(\mathbf{x}))$ reads

$$G(\Phi, \Phi) = \int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}(\delta\phi(\mathbf{x})g(\mathbf{x}, \mathbf{y})\delta\phi(\mathbf{y}) + \delta\pi(\mathbf{x})g^{-1}(\mathbf{x}, \mathbf{y})\delta\pi(\mathbf{y})). \tag{6.18}$$

Notice that these constructions are in line with the conventional definition of Wick symbols in the quantum field theory. Namely, one can check that diagonal blocks $g(\mathbf{x}, \mathbf{y})$ and $g^{-1}(\mathbf{x}, \mathbf{y})$ of the metric (6.18) coincide precisely with the Wick (normal) contractions of the field operators:

$$\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y}) - :\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y}): = \tfrac{1}{2}g^{-1}(\mathbf{x}, \mathbf{y}), \qquad \hat{\pi}(\mathbf{x})\hat{\pi}(\mathbf{y}) - :\hat{\pi}(\mathbf{x})\hat{\pi}(\mathbf{y}): = \tfrac{1}{2}g(\mathbf{x}, \mathbf{y}). \tag{6.19}$$

In terms of holomorphic coordinates

$$a(\mathbf{p}) = \int \frac{\mathrm{d}\mathbf{x}}{(2\pi)^{d/2}}\frac{\mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{x}}}{\sqrt{2\omega(\mathbf{p})}}(\pi(\mathbf{x}) - \mathrm{i}\omega(\mathbf{p})\phi(\mathbf{x})), \tag{6.20}$$

adapted to the Kähler metric (6.18), the Hamiltonian (6.15) takes the form

$$H = \int \mathrm{d}\mathbf{p}\, \omega(\mathbf{p})\bar{a}(\mathbf{p})a(\mathbf{p}). \tag{6.21}$$

Upon quantization the fields $\bar{a}(\mathbf{p})$, $a(\mathbf{p})$ turn to the standard creation/annihilation operators.

6.3. Nonlinear models

Consider now the general Hamiltonian describing a “natural mechanical system”. This is given by the sum of kinetic and potential energies:

$$H = \tfrac{1}{2}h^{ij}(x)p_i p_j + V(x). \tag{6.22}$$

As usual, the kinetic term defines (and is defined by) some metric h on the configuration space of the model $\mathcal{Q}$. Using this metric one can immediately define the Kähler structure on $T^*\mathcal{Q}$ by the general procedure of Section 3, just identifying h with g in the “bare” metric (3.1). This option, however, seems to be not so natural or, at least, it is not the only choice possible. A simple comparison with the case of harmonic oscillator suggests to identify (h_{ij}) with the mass matrix M rather than with a genuine configuration space metric g. The identification for the stiffness matrix K is not so evident and it bears a great amount of ambiguity. The most general ansatz, compatible with the general covariance and harmonic approximation, is given by the following expression:

$$K_{ij}(x) = \nabla_i\nabla_j V(x) + O_{ij}(R, V, \nabla R, \nabla V, \nabla^2 R, \ldots). \tag{6.23}$$

Here ∇_i is a covariant derivative compatible with h and O_{ij} stands for all possible non-minimal terms depending on the curvature R of the metric h and vanishing in the flat limit. The metric g is given by the same expression as for the harmonic oscillator (6.14) with substitution $M = h$; in doing so, the matrix $h^{-1}K$ is supposed to be positive definite.

The field-theoretical counterpart of the above Hamiltonian is known as nonlinear sigma-model. The configuration space of the model consists of all (smooth) maps $\phi : \mathbb{R}^{d,1} \to \mathcal{Q}$ from the $(d+1)$-dimensional Minkowski space to a Riemannian manifold $\mathcal{Q}$ with the metric h_{ij}. In terms of linear coordinates $x = (\mathbf{x}, t)$ on $\mathbb{R}^{d,1}$ and local coordinates ϕ^i on the target manifold $\mathcal{Q}$, any such map is given locally by functions $\phi^i(x)$, which are considered as a set of scalar fields on $\mathbb{R}^{d,1}$. The Hamiltonian of the sigma-model has the standard form $H = T + V$, where

$$T[\phi,\pi] = \frac{1}{2}\int \mathrm{d}\mathbf{x}\, h^{ij}(\phi)\pi_i\pi_j, \qquad V[\phi] = \frac{1}{2}\int \mathrm{d}\mathbf{x}\, h_{ij}(\phi)\partial_\mu\phi^i\partial_\mu\phi^j \tag{6.24}$$

and $\pi_i(\mathbf{x})$ are the momenta canonically conjugate to $\phi^i(\mathbf{x})$, $\{\phi^i(\mathbf{x}), \pi_j(\mathbf{y})\} = \delta^i_j\delta(\mathbf{x}-\mathbf{y})$. The mass matrix associated with the functional of kinetic energy is given by

$$M_{ij}(\mathbf{x},\mathbf{y}) = h_{ij}(\phi(\mathbf{x}))\delta(\mathbf{x}-\mathbf{y}). \tag{6.25}$$

As we have argued above, the simplest choice for the generalized stiffness matrix K is given by the second covariant derivative of the functional $V[\phi]$. So we put

$$K(\delta\phi,\delta\phi)=\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}K_{ij}(\mathbf{x},\mathbf{y})\delta\phi^i(\mathbf{x})\delta\phi^j(\mathbf{y})$$
$$=\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\left[\frac{\delta^2 V[\phi]}{\delta\phi^i(\mathbf{x})\delta\phi^j(\mathbf{y})}-\int \mathrm{d}\mathbf{z}\,\Gamma^l_{ij}(\mathbf{z},\mathbf{x},\mathbf{y})\frac{\delta V[\phi]}{\delta\phi^l(\mathbf{z})}\right]\delta\phi^i(\mathbf{x})\delta\phi^j(\mathbf{y}). \tag{6.26}$$

Here

$$\Gamma^l_{ij}(\mathbf{z},\mathbf{x},\mathbf{y})=\gamma^l_{ij}(\phi(\mathbf{z}))\delta(\mathbf{z}-\mathbf{x})\delta(\mathbf{z}-\mathbf{y}) \tag{6.27}$$

and

$$\gamma^l_{ij}(\phi)=\frac{1}{2}h^{lk}(\phi)\left(\frac{\partial h_{kj}(\phi)}{\partial\phi^i}+\frac{\partial h_{ik}(\phi)}{\partial\phi^j}-\frac{\partial h_{ij}(\phi)}{\partial\phi^k}\right) \tag{6.28}$$

are the Cristoffel symbols of the metrics $M_{ij}(\mathbf{x},\mathbf{y})$ and $h_{ij}(\phi)$, respectively. A simple computation yields

$$K(\delta\phi,\delta\phi)=\int \mathrm{d}\mathbf{x}(h_{ij}(\phi)D_\mu\delta\phi^i D_\mu\delta\phi^j - R_{ikjl}(\phi)\partial_\mu\phi^k\partial_\mu\phi^l\delta\phi^i\delta\phi^j), \tag{6.29}$$

where

$$D_\mu\delta\phi^i=\partial_\mu\delta\phi^i+\gamma^i_{kj}(\phi)\partial_\mu\phi^k\delta\phi^j \tag{6.30}$$

is the covariant derivative and $R_{ikjl}(\phi)$ is the curvature tensor of $h_{ij}(\phi)$. Notice that in the flat limit the second term in the expression (6.29) for the generalized stiffness matrix K vanishes and, in principle, it can be included into the non-minimal terms in (6.23).

7. Conclusion

In this paper we suggest a method of equipping any cotangent bundle over Riemannian manifold with a formal Kähler structure. Having the Kähler structure at hands one can perform an explicitly covariant Wick deformation quantization of a wide class of physical systems along the lines of the general scheme of [10]. With this regard several questions may appear.

First, there is a question of equivalence between the Wick and Weyl deformation quantizations on $T^*\mathcal{Q}$. The vanishing of the first Chern class, shown in Section 5, implies that such an equivalence does take place when $\mathcal{Q}$ is an ordinary manifold. This formal equivalence, however, should be taken cautiously in the field-theoretical context as the equivalence transform from Wick to Weyl symbols may actually diverge because of infinite number of degrees of freedom.[1] So, in order to quantize a realistic physical theories one should work with the Wick symbols from the very beginning.

[1] Upon regularization this transformation can provide no equivalence anymore. Well known example of this phenomenon is offered by the Wick and Weyl quantizations of the free bosonic string, where the phase space is linear and no global obstructions are possible anyway. It seems that in most physically relevant field theories Wick symbols are not equivalent to the Weyl ones.

The second question concerns the fact that the Kähler metric is constructed in a form of power series in momenta, so the $*$-products between generic phase-space functions can be evaluated only as power expansions. This fact does not seem to be an actual restriction for the field/string theory where most physically interesting classical observables are polynomial in momenta. For example, a typical problem is computing the $*$-square of the BRST charge involving first class constraints which almost always are at most squared in momenta.

The third problem to be mentioned in relation to possible field-theoretical applications concerns "the right choice" for the metric on the configuration space of fields. In this paper we have proposed a simple ansatz for such a metric $g = h\sqrt{h^{-1}K}$, where h is the Hesse matrix defined by functional of kinetic energy and K is the matrix of second covariant derivatives (w.r.t. h) of the potential term. Being quite natural and compatible with the experience of free models (where it reproduces the standard Wick symbols in the linear phase space), this choice is by no means unique as one may add any expressions vanishing in the flat limit like the second term in (6.23) or (6.29). One may hope, however, that under certain physical restrictions (like invariance under the global or local symmetries, spatial locality, etc.) all such redefinitions of the metric will lead to formally equivalent quantum theories. On the other hand, the formal equivalence, as has been mentioned above, can be broken in the field theory by regularization of the divergencies, so one can not exclude existence of inequivalent Wick symbols generated by different choices of the configuration space metric.

We are planing to address these questions later.

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